CS 4850

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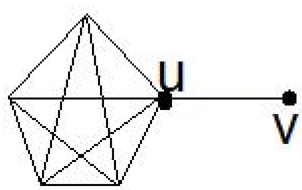
Random Walks on Undirected Graphs

From last time...

Definition 0.1. *The* hitting time h_{uv} *is the expected time to get from vertex* u *to* v.

We note that sometimes adding an edge to the graph can increase hitting time, while other times it can decrease hitting time. It depends on which edge we add. We also observed last time that hitting time is not symmetric: See Figure 1. In this setup, the hitting time h_{vu} is 1, while h_{uv} is much higher because of the clique.



Lemma 0.2. *Consider a chain of vertices numbered* $1, \ldots, n$. *Then the hitting time* h_{1n} *is* $\Theta(n^2)$.

First let's compute:

$$h_{12} = 1$$

For $i \geq 2$, we have

$$h_{i,i+1} = \frac{1}{2}1 + \frac{1}{2}\left[1 + h_{i-1,i} + h_{i,i+1}\right]$$

There is a $\frac{1}{2}$ chance we immediately take the edge to vertex $i+1$, and a $\frac{1}{2}$ chance that we take one step and then are at vertex $i-1$ and we must add in the hitting time from there.

$$2h_{i,i+1} = 2 + h_{i-1,i} + h_{i,i+1}$$

And hence:

$$h_{i,i+1} = 2 + h_{i-1,i}$$

Then the solution has the form $h_{i,i+1} = 2i - 1$ which solves the above recurrence as we check:

$$2i - 1 = 2 + 2(i-1) - 1$$

Let's check this is correct: $h_{23} = 2 + h_{12} = 3$ by direct computation. And our formula gives the same result.

So this is how long it takes to get from vertex i to $i+1$, but how long does it take to get the whole way? We know that

$$h_{1n} = h_{12} + h_{23} + \cdots + h_{n-1,n} = 1 + (3 + 5 + \cdots + 2(n-1) - 1)$$

This is just the sum of an arithmetic series, and hence is quadratic in n. We could also compute this quantity as follows:

$$h_{1n} = \sum_{i=1}^{n-1} h_{i,i+1} = \sum_{i=1}^{n-1}(2i-1) = 2\left(\sum_{i=1}^{n-1} i\right) - \left(\sum_{i=1}^{n-1} 1\right) = 2\frac{(n-1)n}{2} - (n-1) = (n-1)^2$$

Now we ask, how much time do we spend at each vertex on our walk from 1 to n? It turns out that we should expect to spend the most time at vertex 2. Let $t(i)$ be the time spent at vertex i on our walk from 1 to n. We claim:

$$t(i) = \begin{cases} n-1 \text{ if } i = 1 \\ 2(n-i) \text{ if } 2 \leq i \leq n-1 \\ 1 \text{ if } i = n \end{cases}$$

It's clear that $t(n) = 1$. We also know that $t(n-1) = 2$ because there is a one in two chance for us to go to n if we're at $n-1$. For $3 \leq i \leq n-1$ we see $t(i) = \frac{1}{2}[t(i-1) + t(i+1)]$. Our boundary conditions are: $t(2) = t(1) + \frac{1}{2}t(3)$ and $t(1) = \frac{1}{2}t(2) + 1$.

We now have a difference equation on our hands, so it'd be good to know how to solve those.

Difference Equations

Take as an example the difference equation:

$$t(i+1) - 2t(i) + t(i-1) = 0$$

We say this is homogeneous because it's set equal to 0, and that it has constant coefficients because the coefficients are not functions of i. We know that the solution has the form $t(i) = a^i$. So we plug this in to discover a:

$$a^{i+1} - 2a^i + a^{i-1} = 0$$

And in particular, $a^2 - 2a + 1 = 0$. The degree of this equation depends on the degree of the difference equation, in this case 2. Solving this equation, if we have unique roots, gives rise to a basis for a vector space of all the solutions to the recurrence. We can obtain all solutions via linear combinations $xa_1^i + ya_2^i$ of the two roots of the above equation a_1 and a_2.

What if the polynomial has multiple roots? Then for a root of multiplicity two we obtain a solution of the form $t(i) = ia^i$, for multiplicity three $t(i) = i^2a^i$, etc.

Back to Random Walks

Applying this to our particular situation,

$$t(i) = c_1 1^i + c_2 i 1^i = c_1 + c_2 i$$

And $c_1 + c_2 n = 1$, also $c_1 + c_2(n-1) = 2$. So $t(i) = 2(n-i)$ solves this. To check our calculation, the amount of time spent at all the vertices, $\sum_i t(i)$ should be one greater than the expected time spent walking from 1 to n. So we compute:

$$\sum_{i=1}^{n} t(i) = (n-1)+1+2\sum_{i=2}^{n-1}(n-i) = n+2\sum_{i=2}^{n-1} n-2\sum_{i=2}^{n} i = n+2n^2-2\frac{n(n+1)}{2} = n+n^2-n = n^2$$

Which is what we should expect.

Lattices

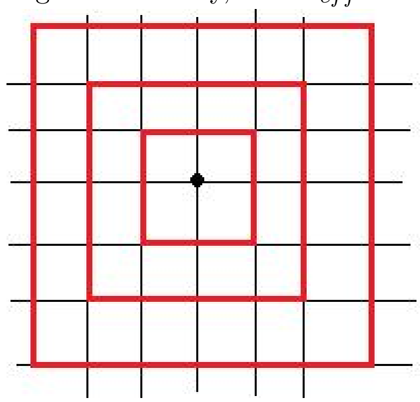
We consider a lattice, first for example will we look at a one-dimensional lattice. We define the *escape probability*, p_{escape}, to be the probability a walk reaches n or $-n$ before returning to origin. We are interested in this probability as $n \to \infty$. It turns out:

$$p_{escape} = \frac{c_{eff}}{c_a}$$

Where c_{eff} is the effective conductance, if we think of this lattices as a network of 1-ohm resistors. We know $r = 1$, $c = 1$, and $c_a = 2$. Therefore $c_{eff} = \frac{2}{n}$. Therefore $\lim_{n\to\infty} c_{eff} = 0$ and therefore $p_{escape} \to 0$.

This problem comes up in other places, for example suppose you're flipping a coin and counting the number of heads minus the number of tails. We know because $p_{escape} = 0$ that in the limit this count is 0.

More interesting is considering what happens in a two-dimensional lattice. We start at the origin, and ask what the probability that we return to the origin on a random walk is. To do this, we first think of the lattice as a network of 1-ohm resistors. Then we short out the resistors as in the figure below, with red resistors shorted out. By shorting these out, we obtain a lower bound on r_{eff}, the effective resistance. If we can show that this lower bound goes to infinity, then $r_{eff} \to \infty$ and hence $c_{eff} \to 0$ and $p_{escape} = 0$.



Every time I go up a level, there are two more resistors. Therefore because the resistors are in parallel we compute the effective resistance as:

$$r_{eff} = \frac{1}{4} + \frac{1}{4 \cdot 3} + \frac{1}{4 \cdot 5} + \cdots = \frac{1}{4}(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots)$$

But this involves the sum of a harmonic series, which diverges. Therefore the escape probability for the two-dimensional lattice is 0. Next time, we'll compute the escape probability in a three-dimensional lattice. Turns out it's not 0.